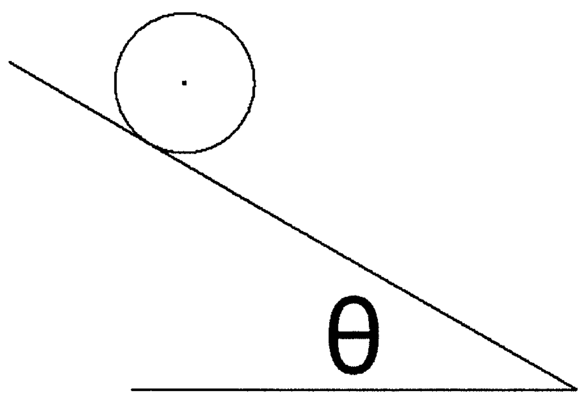
Cart on an Incline with Drag Force

A solid sphere of mass M is released along an incline as indicated in the figure. A drag force described by $F_{drag} = kv$ is the only force that acts to slow the sphere as it rolls along the incline. Let μ be the coefficient of friction between the sphere and the incline.



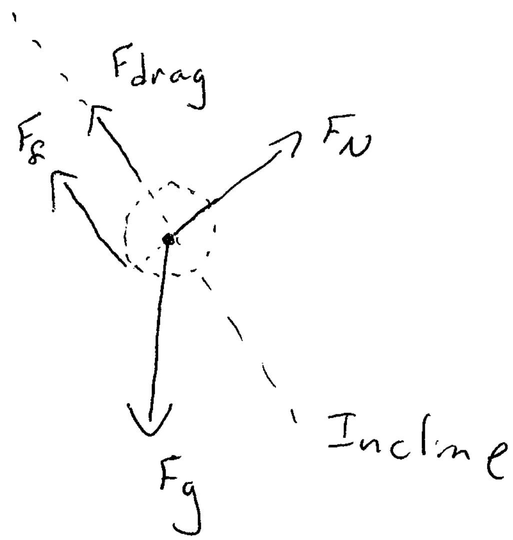
1. Write an equation that describes the terminal velocity the sphere would reach moving down the incline.

2. Write a differential equation that describes the motion of the sphere down the incline. Do not solve.

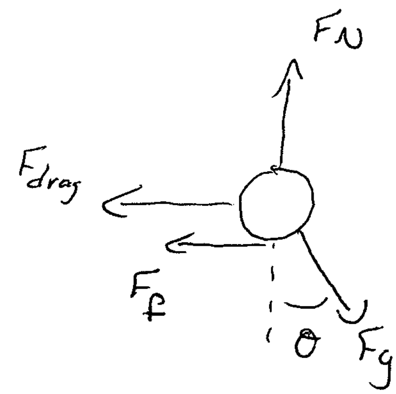
3. Solve the differential equation from the previous part to write an equation that describes the velocity of the sphere along the incline. Assume that the sphere was initially at rest and time started when the sphere was released. In other words; $x_0 = 0$ and $t_0 = 0$

(1)

Step 1: Draw a force diagram



rotated diagram



Step 2: Write an equation using rotated diagram

linear motion

$$\Sigma F_x = ma = -F_f + -F_{drag} + F_g \sin\theta$$

$$\Sigma F_y = 0 = F_N + -F_g \cos\theta$$

~~Step 3:~~

rotation

$$\Sigma \tau = I\alpha = F_f\, r$$

Step 3 : Simplify equations by combining available information

* You want to substitute into ΣF_x to obtain the equation of interest

Use ΣF_y first

$$0 = F_N + -F_g \cos\theta \qquad F_g = Mg$$

$$F_N = Mg \cos\theta$$

$$F_f = uMg \cos\theta$$

Use rotation ($\Sigma\tau$) next

$$\Sigma\tau = I\alpha = F_f r \qquad I = \frac{2}{5}Mr^2$$

$$\alpha = \frac{a}{r}$$

$$\frac{2}{5}\cancel{M}\cancel{r^2}\left(\frac{a}{\cancel{r}}\right) = u\cancel{M}g \cos\theta \cancel{r}$$

$$\frac{2}{5}a = ug \cos\theta$$

now use ΣF_x

$$Ma = -F_f + -F_{drag} + F_g \sin\theta$$

$$Ma = -\mu Mg\cos\theta + -kv + Mg\sin\theta$$

Note from $\Sigma \tau$

$$\mu g \cos\theta = \frac{2}{5}a$$

$$Ma = -M\left(\frac{2}{5}a\right) + -kv + Mg\sin\theta$$

$$\frac{7}{5}Ma = -kv + Mg\sin\theta$$

$$a = -\frac{5k}{7M}v + \frac{5g}{7}\sin\theta$$

$$\boxed{\frac{dv}{dt} = -\frac{5k}{7M}v + \frac{5}{7}g\sin\theta} \quad (2)$$

For (1) recall that terminal velocity means $\Sigma F = 0$

$$\frac{7}{5} M \cancel{a}^{\,0} = -kv + Mg \sin\theta$$

$$kv = Mg \sin\theta$$

$$\boxed{v = \frac{Mg \sin\theta}{k}} \qquad (1)$$

(3)

Step 1: Separate the variables and integrate

$$\frac{dv}{dt} = \frac{-5k}{7M} v + \frac{5}{7} g\sin\theta$$

$$\int_0^v \frac{dv}{\frac{-5k}{7M} v + \frac{5}{7} g\sin\theta} = \int_0^t dt$$

Step 2: Use u-substitution

$$u = -\frac{5k}{7M} v + \frac{5}{7} g\sin\theta$$

$$du = \frac{-5k}{7M} dv$$

limits

$$v \to \frac{-5k}{7M} v + \frac{5}{7} g\sin\theta = b$$

$$0 \to \frac{5}{7} g\sin\theta = a$$

Note: Using a and b to save space.

Step 3:

$$\int_a^b -\frac{7M}{5k}\,\frac{du}{u} = \int_0^t dt$$

$$-\frac{7M}{5k}\ln\left(\frac{b}{a}\right) = t$$

$$\frac{b}{a} = e^{-\frac{5k}{7M}t}$$

$$\frac{-\frac{5k}{7M}v + \frac{5}{7}g\sin\theta}{\frac{5}{7}g\sin\theta} = e^{-\frac{5k}{7M}t}$$

$$-\frac{\cancel{5}k}{\cancel{7}M}v + \cancel{\frac{5}{7}}g\sin\theta = \cancel{\frac{5}{7}}g\sin\theta\, e^{-\frac{5k}{7M}t}$$

$$\frac{k}{M}v = g\sin\theta\left(1 - e^{-\frac{5k}{7M}t}\right)$$

$$\boxed{v = \frac{M}{k}g\sin\theta\left(1 - e^{-\frac{5k}{7M}t}\right)}$$